Mechanics of anisotropic hierarchical honeycombs

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ABSTRACT

Anisotropic hierarchical honeycombs of uniform wall-thickness are constructed by repeatedly replacing each three-edge vertex of a base hexagonal network with a similar but smaller hexagon of the same orientation, and stretching the resulting structure in horizontal or vertical directions to break the isotropy. The uniform overall thickness is then adjusted to maintain the constant average density. The resulting fractal-appearing hierarchical structure is defined by the ratios of replacement edge lengths to the underlying network edge length and also the cell wall angle. The effective elastic modulus, Poisson’s ratio and plastic collapse strength in the principal directions of hierarchical honeycombs were obtained analytically as well as by finite element analyses. The results show that anisotropic hierarchical honeycombs of first to fourth order can be 2.0–8.0 times stiffer and at the same time up to 2.0 times stronger than regular honeycomb at the same wall angle and the same overall average density. Plastic collapse analysis showed that anisotropic hierarchical honeycomb has the larger plastic collapse strength compared to regular hierarchical honeycomb of the same order at certain oblique wall angles. The current work provides insight into how incorporating anisotropy into the structural organization can play a significant role in improving the mechanics of the materials structure such as regular or hierarchical honeycombs, and introduces new opportunities for development of novel materials and structures with desirable and actively tailorable properties.

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1. Introduction

Materials with structural hierarchy over nanometer to millimeter length scales are found throughout Kingdoms Plantae and Animalia. Examples include bones and teeth [1,2], nacre (mother-of-pearl) [3], gecko foot pads [4], Asteriscus (yellow sea daisy) [5], Euplectella sponge [6], wood [2,7] and water repellent biological systems [8]. The idea of using structural hierarchy in engineering structures and materials goes back at least to Eiffel’s Garabit Viaduct and then Tower [9]. More modern examples include polymers [10], composite structures [11–13] and sandwich panel cores [14,15]. The effect of structural hierarchy on mechanical and chemical properties of biological and biomimetic systems has been extensively documented [9–17]. The type and order of the hierarchy and the general organization of these structures play a significant role in their properties and functionality [16,17]. For example, Zhang et al. [16] showed that increasing the level of hierarchy in biological materials increases the toughness but decreases the strength, suggesting that an optimal level of hierarchy could be defined.

Incorporating hierarchy into honeycomb lattice structures has been the focus of a number of studies [15,18–21] and has significance with regard to the application of honeycombs in impact energy absorption and structural protection [22–26], thermal isolation [27] and as the structural core of sandwich panels [28–32]. Recently, a new generation of honeycombs with hierarchical organization was achieved by replacing nodes in the regular honeycomb with smaller hexagons [18,19]. One or two orders of optimized hierarchical refinement offered up to 2 and 3.5 times the in-plane stiffness [18] and almost 2 times the plastic collapse strength [19, 20] of conventional honeycomb with the same mass. Majority of these works address the mechanical and thermal properties of isotropic honeycomb structures. However, there are relatively little investigations on the mechanical properties of honeycomb structure with stretched cells resulting in anisotropic honeycomb structures. The present paper extends these previous works by horizontally/vertically ‘stretching’ or reformulating the underlying hexagonal network prior to the hierarchical refinement steps, so that the developed structure is no longer isotropic (wall thickness is maintained uniform, while being adjusted to have fix overall average density as hierarchy is introduced).

In this work, anisotropic hierarchical honeycombs with various oblique-wall angles are compared to hierarchical conventional honeycombs (with θ = 30°). The stretches not only alter the cell...
wall lengths, but it also changes the oblique wall angle, \( \theta \) (which is equal to 30° in the conventional isotropic honeycomb). Note that uniform stretch leaves oblique cell walls still pointing at the centers of hexagons above and below. Since equal vertical and horizontal stretches would leave the hexagonal geometry undistorted, we 'normalize' the transformation: the length of an oblique cell wall was taken as 30 mm, where \( \theta = 30° \).

In Section 6, results and discussion are demonstrated. In Section 7, conclusions and potential for further performance improvement of hierarchical honeycombs are presented.

2. Fabrication using 3D printing

Fig. 1b shows samples of both zero order and hierarchical regular-hexagon honeycombs with relative density of \( \rho = \rho_s/\rho_l = 0.10 \) and \( l_0 = 20 \text{ mm} \), where \( \rho_s \) is the structural density and \( l_0 \) is the oblique hexagon edge length \cite{18}. These samples were fabricated using 3D printing (Dimensions 3D printer, Stratasys Inc., Eden Prairie, MN). The regular honeycomb has \( t = 1.75 \text{ mm} \); the honeycomb with one level of hierarchy has \( \tau_1 = 0.3 \) and \( t = 1 \text{ mm} \); and that with two-level hierarchy has \( \tau_1 = 0.3 \), \( \tau_2 = 0.12 \), and \( t = 0.75 \text{ mm} \), where \( t \) is the hexagons wall thickness. These were printed as three-dimensional extruded shells from an ABS polymer (acrylonitrile butadiene styrene, elastic modulus \( = 2.3 \text{ GPa} \)).

3. Elastic properties of anisotropic hierarchical honeycombs: analytical modeling

Some geometric constraints on the hierarchically introduced edges must be imposed to avoid interfering with pre-existing members: in a honeycomb with first order hierarchy, \( 0 \leq l_1 \leq l_0/2 \) (Fig. 1a) and thus, \( 0 \leq \tau_1 \leq 0.5 \), where \( \tau_1 = 0 \) denotes the regular honeycomb structure. For second order hierarchy, the two geometrical constraints are \( 0 \leq l_2 \leq l_1 \) and \( l_2 \leq l_0/2 - l_1 \). For uniform wall thickness of \( t \), the relative density of the structure (i.e., area fraction)
can be given as a function of the length ratios and $t/l_0$:

$$\rho_s = \frac{1 + \sin \theta}{3 \sin \theta \cos \theta} \left( 1 + 2 \sum_{n=1}^{\infty} \gamma_n \right) \frac{t}{l_0}$$

(1)

where $\rho_s$ is the material density, $\theta$ is the oblique wall angle, and $n$ is the hierarchical order. This relation is used to adjust $t$, so that various choices of $\theta$ and $\gamma_n$ maintain a fixed relative density. A vertical stretch makes the oblique walls much longer than the horizontal walls, as can be seen by letting $\theta$ approach 0 in the above expressions (Fig. 2a). In following sections, we mainly focus on results and findings. For detailed description of analytical modeling please refer to Appendix A.

3.1. Elastic modulus in principal directions

The effective elastic modulus (to be normalized by the material elastic modulus, $E_s$) is defined as the ratio of average stress ($\tau_x = -F/(3l_0\sin \theta)$, $\tau_y = -P/(l_0\cos \theta)$) and average strain ($\Delta x = -2\Delta \frac{h_1}{l_0\cos \theta}$, $\Delta y = -2\Delta \sin (3l_0\sin \theta)$). $l_0$ is the oblique wall length of the underlying zero-order honeycomb, and $\theta$ is the oblique wall angle. $P$ and $F$ are unit-cell boundary points forces in $x$ and $y$ directions, respectively and $\Delta x$ and $\Delta y$ are the unit-cell boundary point displacements (Fig. 2b and c). The cell walls of thickness $t$ consist of an isotropic elastic material with elastic modulus $E_s$ and Poisson’s ratio $\nu_s$. Using the notation $\gamma_1 = l_1/l_0$, the elastic modulus in $x$ and $y$ directions are finally obtained as

$$E_x = \frac{E_s}{(96 \sin^4 \theta + 596 \sin \theta - 148) \gamma_1^3 + (-48 \sin^2 \theta - 168 \sin \theta + 312) \gamma_1^2 + (-93 \sin \theta - 165) \gamma_1 + (4 \sin^2 \theta + 32 \sin \theta + 28)}$$

and

$$E_y = \frac{4 \cos \theta (\sin^2 \theta + 8 \sin \theta + 7)/(3 \sin^3 \theta)}{192 \sin^7 \theta + 644 \sin \theta - 196) \gamma_1^3 + (-96 \sin^2 \theta - 192 \sin \theta + 336) \gamma_1^2 + (-93 \sin \theta - 165) \gamma_1 + (4 \sin^2 \theta + 32 \sin \theta + 28)}.$$
These expressions, are valid for any allowable $\gamma_1$, in particular for the case when $\gamma_1 = 0$, namely zero order hierarchy. Note also that when $\theta = 30^\circ$, the resulting isotropy means these expressions should be equal for all values of $\gamma_1$. To find the value of $\gamma_1 = l_1/l_0$ which yields the maximum $x$ or $y$ elastic modulus for a given relative density $\rho/\rho_s$, Eq. (1) must be used where $g_2$ is a function of $\gamma_1$, $\gamma_2$ and $\theta$. Differentiating (5) with respect to $\gamma_1 = l_1/l_0$ and $\gamma_2 = l_2/l_0$ at constant density shows that the maximum normalized elastic modulus occurs at $\gamma_1 = l_1/l_0 = 0.3$ and $\gamma_2 = l_2/l_0 = 0.14$.

The Poisson’s ratio $\nu_{xy}$ can then be obtained from $\nu_{xy} = \delta_s / (\gamma_1(\theta)\delta_l)$, which results in

$$\nu_{xy} = \frac{-\cos^2\theta(-112 \sin^2\theta - 564 \sin\theta + 196) \gamma_1^3}{3 \sin^2(192 \sin\theta + 644 \sin\theta - 196) \gamma_1^3} + \frac{(-96 \sin^2\theta - 192 \sin\theta + 336) \gamma_1^3}{3 \sin^2(192 \sin\theta + 644 \sin\theta - 196) \gamma_1^3}$$

(6)

to express $t/l_0$ as a function of $\rho/\rho_s$. For the first order of hierarchy, $n = 1$, this results in $t/l_0 = 3 \sin \theta \cos \theta / (1 + \sin \theta) \cdot \rho/\rho_s(1 + 2\gamma_1)$. (Note that $\tau$ approaches zero when $\theta$ approaches either 0 or 90.) Substituting this into the modulus expressions yields

$$\frac{E_{xy}}{E_x} = \left(\frac{3 \sin \theta \cos \theta}{1 + \sin \theta}\right)^3 \left(\frac{\rho}{\rho_s}\right)^3 f_{xy}(\gamma_1, \theta) \left(\frac{1}{1 + 2\gamma_1}\right)^3$$

(2)

where $E_{xy}$ represents $E_x$ or $E_y$, and $f_{xy}$ correspondingly represents $f_x$ or $f_y$. The analytical tool of MATLAB was used to obtain $\gamma_1$, which maximized Eq. (2). A similar procedure can be used to derive elastic modulus of anisotropic honeycomb structures with two orders of hierarchy (Fig. 2c). The elastic modulus in the principal directions were obtained by finding displacements at location of applied forces, $F$ and $P$. These modulus are function of $\gamma_1$, $\gamma_2$, and $\theta$ and can be presented as: $E_x/E_0 = t^2/l_0^2 \cdot f_x(\gamma_1, \gamma_2, \theta)$ and $E_y/E_0 = t^2/l_0^2 \cdot f_y(\gamma_1, \gamma_2, \theta)$. To find the optimal values of $\gamma_1$, $\gamma_2$, and $\theta$, the constant density ratio $\rho/\rho_s$, the $t/l_0$ term is expressed as a function of density ratio $\rho/\rho_s$. From (1): $t/l_0 = 3 \sin \theta \cos \theta / (1 + \sin \theta) / (1 + 2\gamma_1 + 6\gamma_2) \cdot \rho/\rho_s$. Substituting this into the elastic modulus expressions yields

$$\frac{E_{xy}}{E_x} = \left(\frac{3 \sin \theta \cos \theta}{1 + \sin \theta}\right)^3 \left(\frac{\rho}{\rho_s}\right)^3 f_{xy}(\gamma_1, \gamma_2, \theta) \left(\frac{1}{1 + 2\gamma_1 + 6\gamma_2}\right)^3$$

(3)

For the sake of conciseness, both $E_x$ and $E_y$ are shown in one equation with $f_{xy}$ representing $f_x$ or $f_y$.

3.2. Effective shear modulus and Poisson’s ratio

To fully characterize the linear elastic behavior of horizontally or vertically stretched hierarchical honeycombs, the orthotropic shear modulus $G_{xy}$ and Poisson’s ratio $\nu_{xy}$ should be obtained as a function of $\theta$ and the hierarchical dimension ratios. We applied Castigliano’s second theorem to the full subassemblies of Fig. 2b and c to find shear modulus $G_{xy}$ and Poisson’s ratio $\nu_{xy}$.

The effective shear modulus (normalized by the material shear modulus $G_0$) is defined as the ratio of average shear stress $\tau = -P/(3l_0 \sin \theta)$ to average shear strain ($\epsilon_{xy} = 2\delta_s/(3l_0 \sin \theta) + 2\delta_s/6(l_0 \cos \theta)$). The result can be presented as

$$G_{xy}/G_0 = \left(\frac{3 \sin \theta \cos \theta}{1 + \sin \theta}\right)^3 \left(\frac{\rho}{\rho_s}\right)^3 g_1(\gamma_1, \theta) \left(\frac{1}{1 + 2\gamma_1}\right)^3$$

(4)

where $g_1$ is a complex function of $\gamma_1$ and $\theta$. For a first-order hierarchy the shear modulus occurs when $\gamma_1 = 0.34$, virtually regardless of wall angle $\theta$. However its magnitude depends on $\theta$.

Similarly, for the case of the structure with two hierarchical orders (Fig. 2c), the effective shear modulus can be defined as

$$G_{xy}/G_0 = \left(\frac{3 \sin \theta \cos \theta}{1 + \sin \theta}\right)^3 \left(\frac{\rho}{\rho_s}\right)^3 g_2(\gamma_1, \gamma_2, \theta) \left(\frac{1}{1 + 2\gamma_1 + 6\gamma_2}\right)^3$$

(5)

Performing the same procedure for the second order hierarchy structure of Fig. 2b, we can obtain $\nu_{xy}$ as a function of oblique wall angle $\theta$, $\gamma_1$, and $\gamma_2$. (i.e., $\nu_{xy} = \nu_{xy}(\theta, \gamma_1, \gamma_2)$).

4. Elastic analysis: numerical simulation

To explore the effect of hierarchical order greater than 2, a matrix frame analysis procedure was implemented in MATLAB. Shear and stretching were included in the governing equations for each beam [33]: $d/dx(A \cdot E_x \cdot da/dx) = 0$ for stretching, $d/dx(E_0 \cdot A \cdot G_v \cdot (dv/dx - \phi)) = 0$ for bending, and $d/dx = 1 \cdot d\phi/dx + k_s \cdot A \cdot G_v (dv/dx - \phi) = 0$ for torsion, where $A$ is the cross-sectional area, $l$ is the second moment of area, $E_x$ and $G_v$ are the elastic and shear moduli of the cell wall material, $k_s$ is the shear coefficient (equal to 5/6 for a rectangular cross section [33]), $u$, $v$ are the longitudinal and transverse displacements, and $\phi$ is the beam cross-section rotation about the z-axis. Transformation matrix is calculated for each beam to transform its stiffness matrix to global coordinates and points 3 and 4 were clamped in Fig. 2b and c, respectively. This program was highly efficient and allowed us to systematically change geometry of the hierarchical structure in order to find unique geometry which maximized desired mechanical properties. Compared to the commercial FEA software, the developed program was faster and more flexible. We explored the whole range of $\tau$ to determine the elastic modulus range at each hierarchical order.

5. Plastic collapse strength analysis

In this section, Finite Element simulation is performed to study the in-plane uniaxial plastic collapse strength of hierarchical honeycombs with up to four orders of hierarchy. Using beam element, a subassembly of the structure, as defined in Fig. 2b and c, was modeled in ANSYS. The modeled structure was then meshed using plastic 2D cubic beam element (Beam 23) with plastic, creep, and swelling capabilities. Elastic–perfectly plastic behavior was assigned to the chosen material, and element length was taken as $l_0/500$, where $l_0$ is the length of an oblique wall (short elements are needed to capture localized plastic hinges). The simulations were performed with uniaxial loading separately applied in $x$ and $y$ directions. The elastic modulus, Poisson’s ratio and yield strength of material used in the simulations were $E_0 = 70$ GPa, $\nu = 0.3$ and $\sigma_y = 130$ MPa, respectively. The relative density of the structure used for plastic collapse strength analysis was 0.01.

The modeled subassembly of the structure was subjected to compressive displacement-controlled loading (i.e., on the points.
Fig. 3. (a and b) The elastic modulus bandwidth of specific 1st–4th order hierarchical honeycombs, normalized by the zero-order modulus of isotropic honeycomb (i.e., 30° wall angle) for y and x direction. (c) Surface plot of maximum x-direction and y-direction normalized elastic modulus of zero to fourth order hierarchical honeycomb.

4 and 5 in Fig. 2b and points 5 and 6 in Fig. 2c) with free transverse expansion. Strength was defined as the stress associated with the plateau level in the force–displacement curve. The values of collapse strength in each direction are denoted by $S_{px}$ and $S_{py}$, where $x$ or $y$ denotes the strength direction and $\theta$ is the oblique wall angle.

6. Results and discussion

6.1. Elastic modulus properties

Fig. 3a and b shows the achievable elastic modulus range in the $y$ and $x$ directions as a function of wall angle, for up to 4 hierarchical orders. The computed orthotropic elastic moduli are normalized by the isotropic elastic modulus of zero order unstretched honeycomb (i.e., $E_{0y}$). As can be seen from these two figures, by increasing $\theta$, elastic modulus in the $y$-direction decreases to zero, while normalized $x$-direction modulus ($E_{x}/E_{0y}$) increases to approximately 6. (The opposite behavior is seen as $\theta$ decreases to zero.) These results are explained by the fact that as $\theta$ approaches either 0 or 90, beam thickness reduces and bending stiffness rapidly approaches zero. In contrast axial stiffness contribution increases resulting in an overall finite modulus.

Fig. 3a and b also shows that as hierarchical order is increased, not only the maximum elastic modulus but also the elastic modulus range covered by each additional order increases. Values of $\gamma$ for maximum elastic modulus at each hierarchical order are provided in the figures. These values are approximately independent of $\theta$ and the selection of $x$ or $y$. For third order hierarchy we found $\gamma_1 = 0.27$, $\gamma_2 = 0.13$, $\gamma_3 = 0.06$ and for fourth order we found $\gamma_1 = 0.26$, $\gamma_2 = 0.13$, $\gamma_3 = 0.06$, $\gamma_4 = 0.03$. The trend in $\gamma$ values suggests that when increasing the hierarchical order, the inserted edge length which gives the stiffest structure tends to be half the edge length of the previous hexagons (i.e., $\gamma_1 + \gamma = \gamma_2$).

Fig. 3c compares normalized elastic moduli in the $x$- and $y$-directions ($E_x/E_{0y}$ versus $E_y/E_{0y}$). The $\theta$-constant contours in this figure are shown with solid lines. As can be seen from the figure, by increasing the order of hierarchy ($n$) the covered area of the map increases. Also, for $\theta < 30^o$ the value of $E_x/E_{0y}$ is less than $E_y/E_{0y}$ so that for $\theta < 10^o$ the value of $E_x/E_{0y}$ is negligible compared to $E_y/E_{0y}$ and for $\theta > 60^o$ the opposite is true. Therefore the design region for bi-axial loading should be within the range of $15^o < \theta < 50^o$.

In order to understand the effect of hierarchical structure on the elastic moduli of the structure, the elastic modulus for various hierarchical structures are normalized respect to elastic moduli of the non-hierarchical structure for various wall angle $\theta$. Fig. 4a and b shows the $x$ and $y$ elastic moduli of a first-order hierarchical structure, now normalized by $E_{x,0}$ and $E_{y,0}$, the non-hierarchical structure of the same wall angle, using data shown by a dark line in Fig. 3a and b. Here we divided the stiffness by the stiffness of the zero order value at the same wall angle $\theta$, unlike Fig. 3 in which it is divided by a constant ($E_{0y}$). Surprisingly, the maximum normalized elastic moduli occur in a very narrow range of $\gamma_1 = 0.3147 – 0.3232$ as $\theta$ varies from 0 to 90. At $\gamma_1 = l_1/l_0 = 0.32$, $E_x/E_{y,0} = 2(\sin^2\theta + 8 \sin \theta + 7)/(1.02 \sin^2\theta + 8.12 \sin \theta + 7.02) = 1.98$, which varies less than 1% over the entire
range of wall angle \( \theta \). The maximum elastic modulus in the \( y \)-direction for the first order honeycomb is thus about twice that of non-hierarchical honeycomb with the same density ratio, for all \( \theta \) angles. In contrast, while the \( x \)-direction peak elastic modulus occurs for essentially the same hierarchical structure (same choice of \( \gamma_1 \)), its value does depend significantly on anisotropy, i.e., \( \theta \). Both \( x \) and \( y \) results clearly show that hierarchical refinement with in some range of \( \gamma \) values always increase elastic modulus (i.e., exceed 1.0), regardless of the wall angle.

The data markers in Fig. 4 show the results of computational analyses on the upper half of the subassembly defined in Fig. 2b and c. They represent the MATLAB frame analysis results for four different oblique wall angles and five different \( \gamma_1 \). The results were obtained by clamping point 3 in Fig. 2b and allowing points 1 and 2 to move horizontally due to structural symmetry. Similarly in Fig. 2c, point 4 was clamped and points 1, 2 and 3 were confined to move horizontally. The frame computation results show good correlation with the theoretical results (maximum error \( < 1\% \)). The analytical results were based on using bending energy only. In contrast frame analysis was based on considering all deformation energies, (bending, shear and stretching). The agreement could be justified by considering that the bending energy is dominant energy for small relative density of less than 0.01.

Fig. 4c and d shows the normalized elastic modulus for second order hierarchy in the \( y \)- and \( x \)-directions. The results for the first order hierarchy can be inferred from the plots in Fig. 4a and b at \( \gamma_2 = 0 \). The derivatives of \( (3) \) with respect to \( \gamma_1 = l_1/l_0 \) and \( \gamma_2 = l_2/l_0 \) at constant density show that the maximum normalized elastic modulus in the \( x \)- and \( y \)-directions occur at \( \gamma_1 = l_1/l_0 = 0.29 \) and \( \gamma_2 = l_2/l_0 = 0.153 \), respectively (virtually independent of \( \theta \)), where \( E_y/E_{y,0} \approx 3.52(\rho/\rho_0)^3 \). As can be seen in Fig. 4c, the maximum normalized elastic modulus in \( y \)-direction for \( \gamma_1 = 0.29 \) and for all different oblique wall angles is the same. The maximum elastic modulus in the \( y \)-direction is almost four times greater than the elastic modulus of a regular honeycomb structure with the same density ratio. In contrast, the maximum normalized elastic modulus in the \( x \)-direction depends on the wall angle \( \theta \). There is also a good correlation between frame analysis results and theoretical results (maximum error \( < 1\% \)).

### 6.2. Shear modulus and Poisson’s ratio

Based on the numerical frame analysis in MATLAB, Fig. 5a shows the shear modulus bandwidth \((G_{xy}/G_{xy}^{0})\) of the structure versus oblique wall angle, \( \theta \) for up to 4 hierarchical orders, where \( G_{xy}^{0} \) is the shear modulus of zero order isotropic honeycomb (\( \theta = 30^\circ \)). Fig. 5a shows that by increasing hierarchical order, the possible range of shear modulus increases and the maximum shear modulus is also increases. The oblique wall angle for the maximum shear modulus is approximately \( \theta = 28^\circ \), for all hierarchical orders. Values of \( \gamma \) for maximum modulus at each hierarchical order are virtually independent of \( \theta \). For third order hierarchy this occurs at \( \gamma_1 = 0.27 \), \( \gamma_2 = 0.13 \), \( \gamma_3 = 0.06 \), and for the fourth order at \( \gamma_1 = 0.27 \), \( \gamma_2 = 0.13 \), \( \gamma_3 = 0.06 \), \( \gamma_4 = 0.03 \), \( \gamma_3 = 0.06 \), \( \gamma_4 = 0.03 \). The enhancement in
the normalized shear modulus is noticeable as it increases from 2.59 and 5.04 for 1st and 2nd order hierarchical honeycombs to 7.87 and 10.78 for 3rd and 4th order just as in the previous section on the effect of hierarchical order on the elastic modulus, the trend in $\gamma$ values which maximized the shear modulus tends to be half the edge length of the previous hexagons (i.e., $\gamma_{1,1} = \gamma_{1}/2$). Frame analysis results are also shown which were obtained by applying the same loading conditions as those in the analytical analysis.

The value of $\gamma_{yx}/\nu_{yx}$ is between 0.74 and 0.78 at $\gamma_{1} = 0$ and between 0.35 and 0.38 at $\gamma_{2} = 0.21$, with the minimum value between 0.29 and 0.34 at $\gamma_{2} = 0.17$. As can be seen from these two figures, the curves intersect at a point which correspond to locations in which maximum elastic modulus occurs (i.e., $\gamma_{1} = 0.32$ for the 1st order and $\gamma_{1} = 0.29, \gamma_{2} = 0.135$ for 2nd order). Frame computations in MATLAB are also provided for comparison with the analytical solution. The results show a good agreement between analytical results and the frame analysis. Although the frame analysis considered all deformation modes in the structural components, analytical and frame analysis are in good agreement because bending energy is the dominant energy for low density structures.
6.3. Plastic collapse strength

Fig. 7a and b shows the normalized plastic collapse bandwidth of 1st–4th order hierarchical honeycomb versus oblique wall angle, $\theta$, for $y$- and $x$-directions respectively. Here, the plastic collapse strength is normalized by that of zero order regular honeycomb ($\theta = 30^\circ$). These two figures show that the maximum achievable plastic collapse strength saturates at the 3rd
order for both y- and x-direction. The maximum plastic collapse strength for different hierarchical orders occurs at approximately \( \theta = 28^\circ \) for y-direction and at \( \theta = 45^\circ \) for x-direction and is nearly independent of hierarchical order. The values of \( \gamma \) at which maximum plastic collapse strength occur in each hierarchical order are virtually independent of \( \theta \). For the second order hierarchical structure it was found the maximum x-direction collapse strength occurred when \( \gamma_1 = 0.29 \), \( \gamma_2 = 0.14 \) for y-direction and \( \gamma_1 = 0.29 \), \( \gamma_2 = 0.15 \) for x-direction. For the third order, the maximum strength was achieved at \( \gamma_1 = 0.27 \), \( \gamma_2 = 0.14 \), \( \gamma_3 = 0.07 \) for both x- and y-direction strengths. The trend in \( \gamma \) values show that by increasing the hierarchical order the edge length of newly generated hexagons tends to be half the edge length of the previous hexagons (i.e., \( \gamma_{1+i} = \gamma_1/2 \)). These results show that the collapse behavior of the hierarchical honeycomb can be improved up to the 3rd order by increasing the hierarchical order. However, further refinement does not make any improvement in collapse strength.

Fig. 7c and d shows the normalized x and y direction plastic collapse strengths of a first-order hierarchical structure versus \( \gamma_1 \) for different values of oblique wall angle, \( \theta \). The plastic collapse strength is normalized by that of a zero order honeycomb with same oblique wall angle \( \theta \). The results suggest that the \( \gamma_1 \) values corresponding to the maximum normalized plastic collapse strength in x- or y-direction are not fixed for different oblique wall angles. For y direction, the maximum occurs between \( \gamma_1 = 0.33 - 0.36 \) for the range of oblique beam angle varying between \( \theta = 70^\circ \) and \( 10^\circ \), while for x direction it occurs between \( \gamma_1 = 0.34 \) and 0.35 for the same range of angle. For each curve corresponding to a fixed oblique wall angle, \( \theta \), there are two points with slope discontinuity for the plastic collapse in the y direction (Fig. 7c), and one point for plastic collapse in the x direction (Fig. 7d). These points represent the locations in which the failure mechanism and locations of plastic hinges for different range of \( \gamma \) are shown on top of Fig. 7c and d. For the plastic collapse in the y direction as \( \theta \) decreases, the normalized plastic collapse strength increases. The maximum normalized plastic collapse for \( \theta = 10^\circ \) is equal to 1.37 whereas for \( \theta = 70^\circ \) is equal to 1.08. Fig. 7d shows that \( \theta = 30^\circ \) gives the largest normalized plastic collapse \( S_y/S_y^0 \) which is equal to 1.2 for the 1st order hierarchy structure.

Having the maximum plastic collapse strength at \( \theta = 28^\circ \) for the y direction and at \( \theta = 45^\circ \) for the x direction suggests a map comparing plastic collapse to elastic modulus. Fig. 8a and b shows the maps of normalized collapse strength versus of normalized elastic modulus for zero to 4th order honeycombs for \( \theta = 28^\circ \) in y direction and for \( \theta = 45^\circ \) in x direction, respectively, obtained from the finite element analysis. It should be noted that an nth order hierarchical honeycomb is a special configuration of honeycombs with higher order of hierarchy. For example, a 1st order hierarchical honeycomb is a special configuration of 2nd order hierarchical honeycombs with \( \gamma_2 = 0 \). Thus, the entire colored area in Fig. 8 shows the range of achievable elastic modulus and strength with four orders of hierarchy. Compared to previous data given in [19] for isotropic hierarchical honeycombs, current graphs show a larger range of achievable elastic modulus and plastic collapse strength by stretching the structure in vertical/horizontal direction.

### Table 1

Summary of maximum elastic modulus/strength of anisotropic hierarchical honeycombs.

<table>
<thead>
<tr>
<th>Order</th>
<th>( \varepsilon_0 ) (( \theta = 90^\circ ))</th>
<th>( \varepsilon_1 ) (( \theta = 0^\circ ))</th>
<th>( \varepsilon_{30} ) (( \theta = 28^\circ ))</th>
<th>( \varepsilon_0 ) (( \theta = 45^\circ ))</th>
<th>( \varepsilon_0 ) (( \theta = 28^\circ ))</th>
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<td>6.02</td>
<td>1.02</td>
<td>1.21</td>
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<td>11.98</td>
<td>2.06</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td>( \gamma ) Values</td>
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<td>[0.32]</td>
<td>[0.34]</td>
<td>[0.31]</td>
</tr>
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<td>[0.30, 0.14]</td>
<td>[0.29, 0.15]</td>
</tr>
<tr>
<td>3rd</td>
<td>Property Value</td>
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<td>32.01</td>
<td>7.87</td>
<td>1.79</td>
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<td>( \gamma ) Values</td>
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<td>[0.27, 0.13, 0.06]</td>
<td>[0.27, 0.13, 0.06]</td>
<td>[0.27, 0.14, 0.07]</td>
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<tr>
<td>4th</td>
<td>Property Value</td>
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<td>43.5</td>
<td>10.78</td>
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<td>( \gamma ) Values</td>
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<td>[0.26, 0.13, 0.06, 0.03]</td>
<td>[0.26, 0.13, 0.06, 0.03]</td>
<td>[0.26, 0.13, 0.07, 0.03]</td>
</tr>
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Maximum elastic modulus/strength values of anisotropic hierarchical honeycombs, \( \varepsilon_0 = E_x/E_{30} \), \( \varepsilon_1 = E_y/E_{30} \), \( \varepsilon_{30} = E_y/E_{10} \), \( \varepsilon_{30} = G_{xy}/G_{30} \), \( S_y \) for \( S_y/S_y^0 \) and \( S_y \) for \( S_y/S_y^0 \).
7. Concluding remarks

Mechanical properties of anisotropic hierarchical honeycombs were investigated, where the term ‘anisotropic’ represents the change of honeycomb’s oblique wall angle by uniform horizontal or vertical stretching of the underlying network, and the term ‘hierarchical’ represents the replacement of each three-edge vertex of base hexagon structure with a smaller hexagon of same wall angle. The oblique wall angle of the honeycombs and the relative side lengths of newly generated hexagons are two key parameters which determine the stiffness and strength of the structure, at any fixed relative density.

The effective elastic modulus, Poisson’s ratio and plastic collapse strength of anisotropic hierarchical honeycombs were obtained as a function of dimension ratios and wall angles of the structure. The analytical results for the elastic part were based on Castigliano’s second theorem, and numerical frame analysis was used to validate the analytical approach, and explore higher orders of hierarchy. The results show that a wide range of elastic modulus and strength ratio can be obtained for anisotropic hierarchical honeycombs by varying geometrical parameters. Table 1 summarizes the maximum elastic modulus/strength values of anisotropic hierarchical honeycombs at different hierarchical orders. Increasing the hierarchy order improves the structural performance in terms of its elastic modulus and plastic collapse strength.

Acknowledgments

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Appendix A. Analytical modeling of elastic properties

Elastic modulus in principal directions

we use Castigliano’s second theorem [34] to determine analytically the elastic moduli of first and second order hierarchical stretched honeycombs in their principal directions. The cell walls of thickness t consist of an isotropic elastic material with elastic modulus Es, Poisson’s ratio νs and yield stress σy. Since a conventional hexagonal network extending to infinity has six-fold rotational symmetry, any linear second order tensorial operator (e.g., thermal conductivity) or linear fourth-order operator (e.g., elastic modulus) must be isotropic. However when anisotropy is present, the resulting ‘material’ has only two-fold rotational symmetry, and an orthotropic elasticity tensor. The macroscopic in-plane elastic behavior of an orthotropic material can be described by 5 constants (i.e., Es, Ey, ᵐₑₓ, ᵐₑᵧ and Gₓᵧ), of which four are connected by the reciprocal relation [35]: ᵐₑₓᵧ = ᵐₑᵧₓ. Here Eₓ and Eᵧ are the elastic moduli in the x and y directions normalized by material elastic modulus Es, and ᵐₑₓ and ᵐₑᵧ are orthotropic Poisson’s ratios in the x and y directions, defined as (−strain in second index direction)/ (strain in first index direction) due to uniaxial stress in first index direction. Thus, for network angles other than 30°, instead of two elastic constants governing in-plane deformation, the structure has four.

For the analytical investigation, each cell wall is treated as an Euler–Bernoulli beam, and the stored bending energy is evaluated based on the loading conditions. The modulus is determined from unit-cell boundary point displacements. Following [18], the far-field uniaxial stresses σᵧᵧ = −F/(3lₒ sin θ) or σₓₓ = −P/(lₒ cos θ) were imposed to determine the y- and x-direction elastic moduli, where lₒ is the oblique wall length of the underlying zero-order honeycomb, and θ is the oblique wall angle. Vertical stress is equivalent to applying a vertical force F at every cut point of a horizontal line passing through the mid-points of oblique edges in a row of underlying (i.e., no hierarchy) hexagons (such as points H and H’ in Fig. 1a). Horizontal stress is equivalent to applying a horizontal force P at every cut point of a vertical line passing through the midpoints of oblique edges in a row of underlying (i.e., no hierarchy) hexagons (such as points V, V’, V” and V’’ in Fig. 1a). Due to 180° rotational symmetry, these cut points are moment-free for any choice of uniform remote stress components (i.e., σᵧᵧ, σₓₓ and τ).

Fig. 2b shows a unit cell with 1st order hierarchy subjected to both vertical (σᵧᵧ) and horizontal loading (σₓₓ), however in this section they are analyzed separately. Due to the geometrical and force symmetry, we may apply Castigliano’s method to a reduced subassembly, namely the upper half of the unit cell. For loading in the x- and y-directions, point 4 is subjected only to force −P in the x direction or force −F in the y direction. (For shear loading, to be considered later, the bottom half of the unit cell is also needed, as shown with dotted lines, and non-symmetrical forces as shown are also required.) Due to the structural symmetry, point 3 is clamped for all the analysis [18]. At the subassembly centerline cut points 1 and 2, the y-direction forces acting on the upper subassembly are denoted by N₁ and N₂, and moments are denoted by M₁ and M₂. There can be no horizontal force acting on the upper half-assembly at point 1 because of reflection symmetry about the x axis.

For the first order hierarchy, the elastic bending energy stored in the statically indeterminate subassembly can be expressed as a function of the external force F or P, and the force and moment N₁ and M₁, applied to the structure at point 1: U(F or P, M₁, N₁) = Σall beams \( \int (M^2/2E_1I)\,dx \). Here M is the bending moment at the location s along a cell wall, and I is the beam’s area moment of inertia, which is constant because of the uniform-thickness refinement. To simplify the analysis, we have assumed the normalized structural density to be so low that all macroscopic deformations can be attributed solely to bending of cell walls. (Effectively, axial and shear stiffness are taken to be infinite.)

Assuming zero vertical displacement and zero rotation at points 1, 2, and also along the horizontal beam connecting points 2 and 3 (due to symmetry), N₁ and M₁ can be obtained from \( \partial U/\partial N₁ = 0 \) and \( \partial U/\partial M₁ = 0 \). These two relations allow N₁ and M₁ to be expressed algebraically in terms of F for vertical loading, or P for horizontal loading. The displacements \( \delta_s \) at the point 4 can then be found from \( \delta_s = \partial U(F)/\partial F\big|_{N₁,M₁} \) and \( \delta_s = \partial U(P)/\partial P\big|_{N₁,M₁} \), which allows to obtain strains in the principal directions of structure (for evaluating Poisson’s ratio, both F and P should be applied simultaneously).

For the second order hierarchy, elastic energy stored in the section model of Fig. 2c is obtained as a function of the external force F or P, and unknown reaction forces and moments N₁, M₁, N₂ and M₂ at points 1 and 2: U(F or P, N₁, M₁, N₂, M₂) = Σall beams \( \int (M^2/2E_1I)\,dx \) (where N₁ and M₁ can be written as functions of N₁, N₂, M₁ and M₂ or F and P by means of the equilibrium equations). N₁, M₁, N₂ and M₂ are obtained by invoking the concept of the least work: \( \partial U/\partial N₁ = 0, \partial U/\partial M₁ = 0, \partial U/\partial N₂ = 0, \partial U/\partial M₂ = 0 \). Similar to the first order hierarchy, point 4 is clamped for all the analysis [18].

Shear modulus

To obtain the shear modulus, we imposed far-field shear stress \( \tau \), which gave rise to forces \( P = −3lₒ sin θ \) below the horizontal cut HH’, and forces \( F = −lₒ cos θ \) to the left of vertical cut V’V in Fig. 1a. In a state of pure shear stress, \( P = 3F \cdot tan(θ) \). Since the loading is not symmetric with respect to the unit cell midline, the
full subassembly of Fig. 2b and c must be analyzed to find the stored bending energy.

Using the same procedure as above, the bending energy stored in a first order hierarchical structure can be expressed as a function of the simultaneously applied external forces \( F \) and \( P \):
\[ U = U(F, P). \]
The bending moments along each side of the complete hexagon are determined from a subsidiary analysis in which it is divided at nodes 5 and 6, and then three compatibility conditions are enforced at each of those nodes [18]. Displacements \( \delta_x \) and \( \delta_y \) at point 4 can be found from:
\[ \delta_x = \partial U(F, P) / \partial F \] and
\[ \delta_y = \partial U(F, P) / \partial P. \]

Poisson’s ratio

To obtain the dependence of the orthotropic Poisson’s ratios on the dimension ratio of the hierarchical structure, we again used Castigliano’s second theorem and analyze the upper half of the Fig. 2 subassembly. We determined \( \nu_{xy} \) only, since \( \nu_{yx} = \nu_{xy} \) allows to find \( \nu_{yx} \). Applying far-field stress \( \sigma_{xy} = -F / \partial \theta \sin(\theta) \) in a vertical direction is equivalent to applying a vertical force \( F \) at point 4 of Fig. 2b for first order hierarchy and at point 5 of Fig. 2c for second order hierarchy. Here we apply the dummy horizontal force \( P \) which allows using Castigliano’s theorem to find the horizontal displacement of the node. When \( P \) is zero, the \( x \) and \( y \) displacements of the upper node due to force \( F \) can be expressed as follows:
\[ \delta_x = \partial U / \partial P \bigg|_{N_{1}, M_{1}, P = 0} \] and
\[ \delta_y = U / \partial F \bigg|_{N_{1}, M_{1}, P = 0}, \] where \( U \) is the bending energy stored in the structure due the forces \( F \) and \( P \).

References